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# SANDWICH SEMIGROUPS OF BINARY RELATIONS

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This paper introduces new semigroups of binary relations that arose naturally from investigating the transfer of information between automata and semigroups associated with automata. In particular we introduce a new multiplication on binary relations by means of an arbitrary but fixed "sandwich" relation. R.J. Plemmons and M. West have characterized Green's relations in the usual semigroup of binary relations, and we use these to investigate Green's relations in our semigroups. We give algorithms for constructing idempotents and regular elements in these new semigroups.

## 0. Introduction

In this paper we initiate a study of some new semigroups of binary relations and establish some of their basic properties.

Let $R$ be an arbitrary but fixed binary relation on a finite set $X$ and let $B_X$ denote the set of binary relations on $X$. For $A$ and $B$ in $B_X$ the product of $A$ and $B$, denoted $A*B$, is defined as follows: $(a, b)$ is in $A*B$ if and only if there are $x$ and $y$ in $X$ such that $(a, x)$ is in $A$, $(x, y)$ is in $R$ and $(y, b)$ is in $B$. This product is associative since it is just the usual composition $A \circ R \circ B$ (denoted henceforth as $ARB$). We denote this semigroup by $B_X(R)$ and call it a sandwich semigroup of binary relations with sandwich relation $R$.

To provide some motivation for studying these semigroups we note that semigroups similar to the usual semigroup of binary relations have been the objects of study in the theory of automata. Geller [4, 5, 6] studies semigroups on automata using composition of paths for multiplication as do Fleck, Hedetniemi and Oehmke [3]. However, their work precludes "multiplying" two paths unless the terminal vertex of the first path is the initial vertex of the second path. The idea of "multiplying" two paths by means of a third path connecting the two leads one naturally to sandwich semigroups of binary relations. Relationships between sandwich semigroups of binary relations and automata are studied in [1].

To distinguish properties in $B_X$ from those in $B_X(R)$ we precede those in $B_X(R)$ by an $R$, as in $R$-regular. In Section 1 general properties of $B_X(R)$ and relationships to $B_X$ are established. Given $R$ we characterize $R$-regular elements in Section 2, give an algorithm for determining if an element $A$ is $R$-regular and give an algorithm for finding an inverse if $A$ is $R$-regular. Based on these results we give a characterization of $R$-idempotents.

## 1. Preliminaries

We say a relation $A$ is *$R$-transitive* if an only if $A*A$ is contained in $A$ and we say $A$ is an *$R$-equivalence* relation if and only if it is reflexive, symmetric and $R$-transitive. If $R$ is an equivalence relation, then it is an $R$-equivalence relation since $R^2 \subseteq R$ implies $R^3 \subseteq R$. If $I \subseteq R$ and $A$ is an $R$-equivalence relation, then $ARA \subseteq A$, and since $I$ is in $A$ and $R$, we have $A \subseteq ARA$. Thus, $A = ARA$ and $A$ is an *$R$-idempotent*. Further, the *$R$-join*, $A \vee_R B$, of two $R$-equivalence relations is the smallest $R$-equivalence relation containing $A$ and $B$.

**Lemma 1.1.** *Let $A$, $R$ and $B$ be $R$-equivalence relations on a set $X$ and $A*B = B*A$. Then $A*B$ is an $R$-equivalence relation on $X$ and is the $R$-join of $A$ and $B$.*

**Proof.** We show $A*B$ is contained in $A \vee_R B$ and $A*B$ is an $R$-equivalence. Then since $A$ and $B$ are in $A*B$ we have $A*B = A \vee_R B$.

Let $(a, b)$ be in $A*B$. Then $(a, b) = (a, x)(x, y)(y, b)$ for some $(a, x)$ in $A$, $(x, y)$ in $R$ and $(y, b)$ in $B$. Now $(a, x)$ and $(y, b)$ are in $A \vee_R B$, but $A \vee_R B$ is $R$-transitive. Hence, $(a, b) = (a, x)(x, y)(y, b)$ is in $A \vee_R B$ and $A*B \subseteq A \vee_R B$.

$A*B$ is reflexive since $I \subseteq A \subseteq ARB$ and $R$-transitive since

$$(A*B)*(A*B) = AR(BRA)RB = (ARA)R(BRB) = ARB = A*B.$$

We show $A*B$ is symmetric. Let $(a, b)$ be in $ARB$. Then $(a, b) = (a, x)(x, y)(y, b)$ as above. Thus $(b, a) = (b, y)(y, x)(x, a)$ which is in $BRA = ARB$.

We now recall Green's relations for a semigroup $S$. If $a$ and $b$ are in $S$, then $a\mathscr{L}b$ $(a\mathscr{R}b)$ means $a$ and $b$ generate the same principal left (right) ideal of $S$. That is $S^1a = S^1b$ $(aS^1 = bS^1)$ where $S^1$ is $S$ if $S$ has an identity element and otherwise is $S$ with an identity element adjoined. We write $a\mathscr{J}b$ if $S^1aS^1 = S^1bS^1$, that is $a$ and $b$ generate the same two-sided ideal. We also have the relations $\mathscr{H}$ and $\mathscr{D}$ where $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$ and $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. We now look at Green's relations for $B_X(R)$ where given relations $A$ and $B$ we write $A\mathscr{L}_R B$ to denote the $\mathscr{L}$ relation in $B_X(R)$ and $A\mathscr{L}B$ to denote the $\mathscr{L}$ relation in $B_X$. Since Green's relations for $B_X$ have been characterized by Plemmons and West [7], we have the following lemma.

**Lemma 1.2.** (i) *$A\mathscr{L}_R B$ implies $A\mathscr{L}B$.*
(ii) *$A\mathscr{R}_R B$ implies $A\mathscr{R}B$.*
(iii) *$A\mathscr{H}_R B$ implies $A\mathscr{H}B$.*
(iv) *$A\mathscr{J}_R B$ implies $A\mathscr{J}B$.*
(v) *$A\mathscr{D}_R B$ implies $A\mathscr{D}B$.*

The following computational rules in the semigroup $B_X(R)$ follow immediately from those in the known semigroup $B_X$. Let $A$, $B$ and $C$ be in $B_X(R)$. Then

(i) $A \subseteq B$ implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$.

(ii) $A * (\bigcup_{i \in I} B_i) = \bigcup_{i \in I} A * B_i$.

(iii) $A * (\bigcap_{i \in I} B_i) \subseteq \bigcap_{i \in I} A * B_i$.

In this and subsequent sections we frequently let the set $X$ be $X = \{x_1, x_2, \ldots, x_n\}$ and denote by $x_i A$ the set $\{y_1, y_2, \ldots, y_k\}$ of elements in $X$ such that for all $j \in \{1, \ldots, k\}$, $(x_i, y_j)$ is in $A$.

We also use Boolean matrices to represent relations. If $A$ is a relation in $B_X(R)$ where $X$ is as above, then it is represented by the matrix in which the $(i, j)$ entry is 1 if $(x_i, x_j)$ is in $A$ and 0 otherwise. Here $x_i A$ is determined by the nonzero entries in the $i$th row. Furthermore, if row $i$ of $A$ is the componentwise Boolean sum of rows $p_1, \ldots, p_t$ of $A$, then we have $x_i A = \{x_{p_1}, \ldots, x_{p_t}\} A$ where

$$\{x_{p_1}, \ldots, x_{p_t}\} A = x_{p_1} A + \cdots + x_{p_t} A.$$

In computations if $(x_i) AB = \{x_{p_1}, \ldots, x_{p_t}\} B$ where $\{x_{p_1}, \ldots, x_{p_t}\} = x_i A$, then we may sometimes write $\{x_{p_1}, \ldots, x_{p_t}\} B = x_{p_1} B \cup \cdots \cup x_{p_t} B$ instead of $x_{p_1} B + \cdots + x_{p_t} B$.

## 2. Regular elements in $B_X(R)$

As usual, $B_X(R)$ is said to be regular if for each $A$ in $B_X(R)$ there is a $B$ in $B_X(R)$ such that $A * B * A = A$ or $ARBRA = A$. Thus we note that regularity in $B_X(R)$ implies regularity in $B_X$. The converse is not true as a later example will illustrate.

The following lemma gives some relationships between $R$-regularity and $I$-regularity where $I$ is represented by the identity matrix. The proof is straightforward and is omitted.

**Lemma 2.1.** *Let $A$ be an element of $B_X(R)$.*

(i) *If $R$ is invertible, $R$-regular is equivalent to $I$-regular.*

(ii) *If $A$ is $R$-regular, then $AR$ and $RA$ are $I$-regular.*

(iii) *$A$ is $I$-regular if and only if $A$ is $R'$-regular for some $R'$.*

We also recall that if $a$ is a regular element of a semigroup $S$ such that $axa = a$, then $ax$ and $xa$ are idempotents. In view of this, the relationships between regularity and idempotents given in the following lemma are not surprising. We use the terms $R$-idempotent and $I$-idempotent in the same manner as $R$-regular and $I$-regular.

**Lemma 2.2.** *Let $A$ be an element of $B_X(R)$.*

(i) *If $A$ is $R$-regular, then $A$ is an $R'$-idempotent for some $R'$.*

(ii) *If $A$ is an $R$-idempotent, then $A$ is $I$-regular.*

(iii) $\{A\colon A$ *is $I$-regular*, $A = ABA\} = \{A\colon A$ *is a $B$-idempotent*$\}$.

(iv) *If $A$ is $R$-regular and $AR$ is an $I$-idempotent, then $A$ is an $R$-idempotent.*

(v) *If $A$ is an $R$-idempotent and $AR$ is $I$-regular, then $A$ is $R$-regular.*

(vi) *Let $A$ be an $R$-idempotent. Then $AR$ is $I$-regular if and only if $A$ is $R$-regular.*

(vii) *$A$ is $R$-regular if and only if there exists $B$ and $C$ in $B_X(R)$ such that $ARBR$ is an $I$-idempotent and $ARBRC = A$.*

Let $A$ be in $B_X(R)$ and assume there is a $B$ in $B_X(R)$ such that $A*B = A$ and for this $B$, a $C$ such that $C*A = B$. Then $A = A*B = A*C*A$ and $A$ is $R$-regular. Conversely, if $A$ is $R$-regular, $B$ and $C$ will exist. That is, if $A = A*D*A$, then let $C = D$ and $B = D*A$. The following two propositions show when for an arbitrary $A$ we have $B$ and $C$ as above. Furthermore, the propositions are constructive and so we have an algorithm to determine whether or not a given $A$ in $B_X(R)$ is $R$-regular and if $A$ is $R$-regular we can find an inverse.

If $A$ is in $B_X(R)$ and $x$ is in the domain of $AR$, then we let $I_x$ denote the following set: $I_x = \{y\colon (y, x)$ is in $AR\}$. Thus, scan the $x$th column of $AR$ and $y$ is in $I_x$ if there is a nonzero entry in the $(y, x)$ position. If $x_i$ is in $xAR$, then the $(x, x_i)$ entry of $AR$ is nonzero and so $x$ is in $I_{x_i}$.

**Proposition 2.3.** *Let $A$ be in $B_X(R)$ such that if $x$ is in the domain of $A$, then $x$ is in the domain of $AR$. Let $W_{xAR} = \bigcup_i (\bigcap_y yA)$ for all $x_i$ in $xAR$ and $y$ in $I_{x_i}$. Then $W_{xAR} = xA$ if and only if there is a $B$ in $B_X(R)$ satisfying $A*B = A$.*

**Proof.** Assume $W_{xAR} = xA$. We will define $B$ such that $A*B = A$. If $xA = \emptyset$, then let $xB = \emptyset$. If $xA \neq \emptyset$, then define $xB = \bigcap_{y \in I_x} yA$. We claim $A*B = A$. If $xA = \emptyset$, then $(x)A*B = \emptyset$, and so $(x)A*B = xA$. If $xA \neq \emptyset$, then $xAR \neq \emptyset$ and so let $xAR = \{x_1, \ldots, x_m\}$. Then

$$x(A*B) = (xAR)B = \{x_1, \ldots, x_m\}B = x_1B \cup \cdots \cup x_mB$$

$$= \bigcup_{x_i \in xAR} x_iB = W_{xAR} = xA.$$

Observe if $x_iB = \emptyset$ for some $i$, the equality is unchanged. If $x_iB = \emptyset$ for all $x_i$ in $xAR$, then on one hand $(xAR)B = \emptyset$ and on the other

$$\emptyset = \bigcup_{x_i \in xAR} x_iB = W_{xAR} = xA.$$

Hence, in all cases $A*B = A$.

Conversely, assume there is a $B$ such that $A*B = A$. If $xA = \emptyset$, then $W_{xAR} = \emptyset$ and so $W_{xAR} = xA$. Thus, assume $xA \neq \emptyset$ and let $z$ be in $xA$. Then $z$ is in $x(A*B) = (xAR)B$. Therefore, there is $x'$ in $xAR$ such that $z$ is in $x'B$. For each

$y$ such that $x'$ is in $yAR$ (that is $y$ is in $I_{x'}$) then $x'B$ is in $yARB = yA$. Thus, $z$ in $x'B$ implies $z$ is in $yA$ for each $y$ in $I'_x$ and so $z$ is in $W_{xAR}$. On the other hand, let $z$ be in $W_{xAR}$. Then $z$ is in $\bigcap_y yA$ ($y$ in $I_{x_i}$) for some $x_i$ in $xAR$. Thus $z$ is in $yA$ for all $y$ in $I_{x_i}$. But, $x$ is in $I_{x_i}$ so $z$ is in $xA$.

**Proposition 2.4.** *Let $A$ and $B$ in $B_X(R)$ for some $R$. Then there is a $C$ in $B_X(R)$ such that $C * A = B$ if only if for each $x$ in $X$ there is a subset $K_x$ of $X$ satisfying $(K_x)RA = xB$.*

**Proof.** If $C * A = B$, then $xCRA = xB$. Hence, let $K_x = xC$. Conversely, for $x$ in $X$ define $xC = K_x$.

Combining Propositions 2.3 and 2.4 we have the following characterization of $R$-regular elements.

**Theorem 2.5.** *Let $A$ be in $B_X(R)$ and if for all $x$ in $X$ we have $W_{xAR} = xA$, then we let $B$ be the relation such that $A * B = A$. $A$ is $R$-regular if and only if for all $x$ in $X$, $W_{xAR} = xA$ and there is a set $K_x$ such that $(K_x)RA = xB$.*

Thus, to determine if $A$ is $R$-regular we show $W_xAR = xA$ for all $x$ in $X$ and define $B$ as in the proof of Proposition 2.3. After verifying the existence of a set $K_x$ for each $x$ in $X$, we find an $R$-inverse, $C$, for $A$ by $xC = K_x$ for each $x$ in $X$.

**Example 2.6.** Let

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Straightforward computations show $W_{xAR} = xA$ for all $x$ in $X$. From the proof of Proposition 2.3 we find $x_1B = \{x_2\} = x_3B$ and $x_2B = \{x_1, x_2\}$ and so

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Row 1 of $B$ is row 1 of $RA$ so we let $K_{x_1} = \{x_1\}$ and $x_1C = \{x_1\}$. Similarly, $x_2C = \{x_2, x_3\}$ and $x_3C = \{x_1\}$. Since $A = A * C * A$ then $C$ is an inverse for $A$. Since rows 2 and 3 of $RA$ are the same, we could also have defined $x_2C = \{x_2\}$ or $x_2C = \{x_3\}$.

**Theorem 3.1.** *Let $A$ be in $B_x(R)$. $A$ is an $R$-idempotent if and only if $AR$ is an $I$-idempotent and for all $x$ in the domain of $AR$, $W_{xAR} = xA$.*

**Proof.** If $A = ARA$ then $AR = ARAR$. In Proposition 2.3 let $B = A$ and we have $W_{xAR} = xA$ since $A * B = A$.

Conversely, $W_{xAR} = xA$ implies there is a $B$ in $B_x(R)$ such that $A * B = A$. Then $A = A * B = ARB = AR(ARB) = ARA = A * A$.

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